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Diffraction of Scalar Waves By a Circular Aperture

J. BAZER and A. BROWN

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J. Bazer and A. Brown

Dr. Werner W. Gerbes
Dr. Werner Gerbes
Contract Monitor

J. Bazer
J. Bazer

A. Brown
A. Brown

Sidney Borowitz
Sidney Borowitz
Acting Director

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Abstract

The problems of the diffraction of scalar waves by a circular aperture in a perfectly soft and in a perfectly rigid infinite planar screen are treated, assuming axially-symmetric excitation. New integral representations of the solutions are presented which, on the shadow side of the screen, are expressed as follows:

$$\phi_{1,2} = \int_{-1}^{1} \left\{ \exp\left[ikR^{+}(x,y,z;t)\right]/R^{+}(x,y,z;t) \right\} f_{1,2}(t)\,dt.$$

Here, the suffixes 1 and 2 refer, respectively, to the soft and rigid-screen problems; k is the wave number:

$$R^{+}(x,y,z;t) = \sqrt{\rho^{2} + (z + iat)^{2}}, \text{ where } \rho = \sqrt{x^{2} + y^{2}};$$

$z = 0$ defines the plane of the screen; 'a' is the radius of the aperture, centered at the origin; and $\rho = 0$ defines the axis of symmetry. The unknown functions $f_1(t)$ and $f_2(t)$ are, in order, required to be odd and even analytic functions of t, regular in a neighborhood of the unit disc $0 \leq |t| \leq 1$. These integral representations are designed to satisfy automatically the time-reduced wave equation, the radiation condition and the boundary conditions, and to have the appropriate edge behavior; $f_1(t)$ and $f_2(t)$ need not be known explicitly. A relatively brief and elementary calculation reveals the f's to be solutions of Fredholm-type integral equations of the second kind whose kernels are small when ka is small. The integral equations guarantee the existence of the desired types of f's whenever the exciting fields are suitably regular functions of ρ in a neighborhood of the aperture, and, moreover, lead to approximate expressions for the f's when ka is small. These approximate f's are employed to calculate the aperture fields, the far fields, the transmission coefficients and the edge behaviors. The results thus obtained are in complete agreement with known results and, for some of these quantities, are more accurate, in that higher powers of ka are included. In addition, in the rigid-screen problem, for all positive values of ka, we determine the form of the edge behavior and give a simple proof of the unique existence of the solution.

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1. Introduction

We are concerned with the problems of diffraction by a circular aperture in a perfectly soft and in a perfectly rigid, infinite, planar screen*. Our attention is focused mainly on the low frequency end of the spectrum, where the quantity $\alpha = ka$ (k is the wave number, a is the aperture radius) is small. We treat in detail only the case of normally-incident, plane-wave excitation, although the method is applicable to arbitrary axially-symmetric excitation and may, in fact, be extended to cover arbitrary angularly-dependent excitation**. New integral representations $\mathcal{R}_1(f_1)$ and $\mathcal{R}_2(f_2)$ of the solutions of $\mathcal{P}_1$ and $\mathcal{P}_2$ are presented [f_1 and f_2 are unknown functions - see equations (3.1), (3.2) and (5.1)]. These representations have the virtue of being both simple and of leading directly to Fredholm integral equations of the second kind for f_1 and f_2. We use the fact that the kernels of the Fredholm equations are small when α is small to calculate the first several terms of the Neumann series of f_1 and f_2. These expressions are, in turn, employed to calculate approximate expressions for the aperture fields, the far fields, the transmission coefficients and the edge behaviors. The results thus obtained are in complete agreement with known results obtained by other means - such as have been summarized, for example, by C.J. Bouwkamp in his excellent review article [1]. Moreover, by including higher powers of α, we extend the accuracy of some of the results quoted by Bouwkamp. In addition, in $\mathcal{P}_2$ we give, for all values of $\alpha > 0$, the form of the edge behavior and a simple proof of the unique*** existence of the solution.

---

* The soft-screen problem, or first boundary-value problem, will hereafter be referred to as $\mathcal{P}_1$ and the rigid-screen problem, or second boundary-value problem, as $\mathcal{P}_2$.

** This generalization will be treated in a forthcoming paper by J. Bazer and H. Hochstadt.

*** Establishing the uniqueness for both $\mathcal{P}_1$ and $\mathcal{P}_2$ is a relatively trivial matter (cf. pp. 19-20).

The integral representations mentioned above form the core of our paper. They are the natural generalizations of similar well-known solutions of the potential-theoretic analogs of ρ_1 and ρ_2 such as may be found in the report [2] by S. N. Karp (our original source) and in the book [3]* by Green and Zerna. Several authors, among whom are W. Magnus [4], N. Chako [5], D. S. Jones [6] N.I. Akhiezer and A.N. Akhiezer [7], B. Noble [8]** and R.C. MacCamy and A.E. Heins [9]*, have succeeded in reducing aperture and disc problems to solving Fredholm integral equations of the second kind with kernels that are small when α is small***. Our integral equations are most closely related to one first obtained by D.S. Jones [6]****. Although the general form of our inhomogeneous term is simpler, the kernels have fundamentally the same form. Owing mainly to the simple form of the initial integral representations, our derivations of the Fredholm equations are shorter and more elementary than any by the authors mentioned above. In each of our representations, the field is related to the solution of the associated Fredholm equation by means of a single integral, and the derivation of the Fredholm equation involves only the inversion of a single convolution of the solution. In the work of Jones, and of MacCamy and Heins*****, for example, double integrals replace the single integrals, and successive convolutions replace the single convolutions[*]. In addition, we can verify, almost at a glance, that the total fields satisfy the boundary conditions, as well as the wave equations and the radiation condition; it is not necessary to revert to more primitive representations as in some of the above approaches. Furthermore, the general form of the fields near the edge can be ascertained by a simple calculation.

---

* We are indebted to B. Noble for bringing references [3] and [9] to our attention.

** Using the so-called dual integral-equation approach, B. Noble finds it possible to treat both slit and aperture problems simultaneously.

*** We refer the reader to C.J. Bouwkamp's review [1] for a discussion of other methods.

**** In reference [6], Jones treats the rigid disc problem which, by virtue of Babinet's principle, is equivalent to our ρ_1.

***** The approach of these authors also leads to a Jones—type integral equation.

It is true that our approach depends essentially on an *ansatz*, whereas the several approaches mentioned above employ possibly more familiar points of departure. It is our belief that the transparent nature of the representations and the elementary character of the subsequent analysis more than make up for this seeming defect. Moreover, preliminary calculations reveal that the scope of the method is by no means limited to scalar or to three-dimensional aperture problems. The same is undoubtedly true of the other approaches; however, the simplicity of the present method may well bring within reach orders of approximation that, for all practical purposes, are inaccessible to more cumbersome methods.

---

[*] It is for this reason that the general form of the inhomogeneous terms of our Fredholm equations is simpler than Jones'.

2. Formulation of the Problems $\mathcal{P}_1$ and $\mathcal{P}_2$.

Let $\mathcal{A}$ be a circular aperture of radius a in a rigid screen $\mathcal{S}$ and let $\mathcal{A} + \mathcal{S}$ coincide with the (x,y)-plane, the origin being located at the center of $\mathcal{A}$ (see Figure 1).

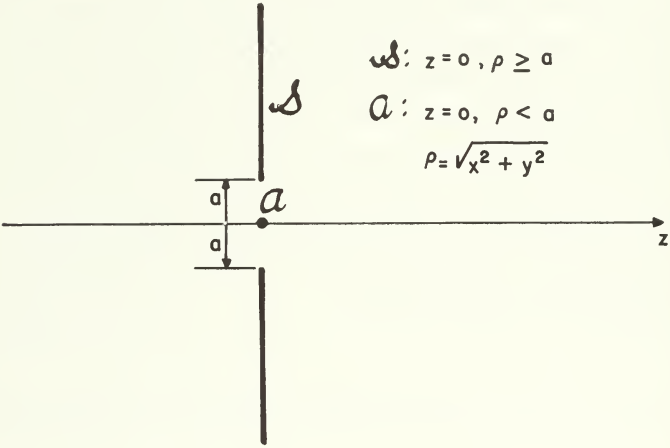
$\mathcal{S}$: z = 0 , $\rho \geq a$
$\mathcal{A}$: z = 0, $\rho < a$
$\rho = \sqrt{x^2 + y^2}$
$\mathcal{S}$
$\mathcal{A}$
a
a
z

FIGURE 1.

A cross-sectional view of the diffracting screen. $\mathcal{S}$ is the screen, $\mathcal{A}$ is the aperture, of radius a, centered at the origin of coordinates. $\mathcal{S} + \mathcal{A}$ is the (x,y)-plane.

Then, the solution u_1 of the first boundary-value problem $\mathcal{P}_1$ may be expressed as follows [cf. reference [1], p.38]*:

$$u_1(\rho,z) = \begin{cases} u^o(\rho,z) - u^o(\rho,-z) + \phi_1(\rho,-z) , & z < 0 , \\ \phi_1(\rho,z) , & z \geq 0 . \end{cases} \tag{2.1}$$

Here, ρ is defined by

$$\rho = \sqrt{x^2 + y^2} \quad ; \tag{2.2}$$

$u^o(\rho,z)$ denotes the axially-symmetric exciting wave incident from the left (in the region $z < 0$); $\phi_1(\rho,z)$ is defined in the half-space $z \geq 0$, and has the following properties:

(2.3)

(i) ϕ_1 is a solution of the wave equation $\Delta\phi_1 + k^2\phi_1 = 0$, when $z > 0$;

(ii) $\phi_1 = 0$ on $\mathcal{S}$;

(iii) ϕ_1 satisfies the Sommerfeld radiation condition at infinity;

(iv) $\dfrac{\partial\phi_1}{\partial z} = \dfrac{\partial u^o}{\partial z}$ in $\mathcal{A}$;

(v) ϕ_1 is everywhere finite ;

(vi) $\nabla\phi_1$ is quadratically integrable over any domain in 3-space, including domains having some or all of the points of the circular edge as boundary points.

---

* In view of the assumed axial symmetry of the incident excitation, all field quantities may be assumed to depend on $\rho = \sqrt{x^2 + y^2}$ and z alone.

The corresponding solution u_2 of the second boundary-value problem $\mathcal{P}_2$ may be expressed as follows:

$$(2.4) \quad u_2(\rho,z) = \begin{cases} u^o(\rho,z) + u^o(\rho,-z) - \phi_2(\rho,-z), & z \leq 0, \\ \phi_2(\rho,z), & z \geq 0. \end{cases}$$

Here, $\phi_2(\rho,z)$ is defined for $z \geq 0$ and possesses all the properties listed above except (ii) and (iv), which must be replaced by

$$(2.5) \quad \begin{aligned} &(ii)' \quad \frac{\partial \phi_2}{\partial z} = 0 \text{ on } \mathcal{S}; \\ &(iv)' \quad \phi_2 = u^o \text{ in } \mathcal{A}. \end{aligned}$$

Let us write

$$(2.6) \quad u_1^o(\rho) = \frac{\partial u^o}{\partial z} \text{ in } \mathcal{A},$$

and

$$(2.7) \quad u_2^o(\rho) = u^o \quad \text{in } \mathcal{A}.$$

It is henceforth assumed that $u_j^o(\rho)$, $j = 1,2$, is an even, analytic function of ρ whose analytic continuation is regular at least in a (closed) circular domain of radius $a + \Delta, \Delta > 0$, centered at $\rho = 0$. It is also assumed that the Laplace transform of $u_j^o(\rho)$, $j = 1,2$, ρ real, exists[*]. In the special case of normally incident, plane-wave excitation[**]

$$(2.8) \quad u^o = e^{ikz},$$

---

[*] For sufficient conditions, we refer the reader to reference [10].

[**] Time dependence of the form $\exp(-i\omega t)$ is assumed throughout.

so that

$$u_1^o(\rho) = ik , \tag{2.9}$$

and

$$u_2^o(\rho) = 1 . \tag{2.10}$$

These functions clearly have the required properties.

3. The Basic Integral Representation for ϕ_1

We require that ϕ_1 have an integral representation of the form

$$\phi_1 = \int_{-1}^{1} \frac{e^{ik\,R^+(\rho,z;t)}}{R^+(\rho,z;t)}\, f_1(t)dt \tag{3.1}$$

or, equivalently, of the form

$$\phi_1 = \int_{0}^{1} \left\{ \frac{e^{ik\,R^+(\rho,z;t)}}{R^+(\rho,z;t)} - \frac{e^{ik\,R^-(\rho,z;t)}}{R^-(\rho,z;t)} \right\} f_1(t)dt. \tag{3.2}$$

Here, $f_1(t)$ is required to be an odd function of t, and

$$R^{\pm}(\rho,z;t) = \sqrt{\rho^2 + (z \pm iat)^2} , \tag{3.3}$$

where the upper and lower signs on the left correspond, respectively, to the upper and lower signs in the right member. Since the regularity of $u_1^o(\rho)$ in the (closed) region defined by $0 \leq |\rho| \leq a + \Delta$ (see the concluding paragraphs of Section 3.) ultimately implies that $f_1(t)$ is regular in the circular region $\mathcal{C}_\Delta$ defined by $0 \leq |t| \leq 1 + \Delta/a$, we may, without loss of generality require $f_1(t)$ to be regular in $\mathcal{C}_\Delta$ at the outset. The branches of the square roots in (3.3) are fixed by requiring that

$$(3.4) \qquad \lim_{z\downarrow 0} R^{\pm}(\rho,z;t) = +\sqrt{\rho^2 - a^2t^2} \qquad \text{when } \rho^2 > a^2t^2 .$$

This specification immediately implies that

$$(3.5) \qquad \lim_{z\downarrow 0} R^{\pm}(\rho,z;t) = \pm\, i\sqrt{a^2t^2-\rho^2} \qquad \text{when } \rho^2 < a^2t^2 .$$

At points in the aperture, the only points in physical space where the integrands of equations (3.1) and (3.2) may be singular, ϕ_1 and its first and higher order derivatives may be defined as the limits approached by ϕ_1 in $z > 0$ as z approaches zero. The actual limits can be derived using the regularity properties of $f_1(t)$ in $\mathcal{C}_\Delta$ and function-theoretic arguments such as are found in reference [11]. An illustration of the kind of argument required is given in Appendix I, where the form of the edge behavior is determined. Apart from this material, we shall not justify the several limiting processes used in the subsequent analysis. It is enough to say that the limits agree with those arrived at on the basis of purely formal considerations. Finally, from equation (3.1), we see that ϕ_1 may be described as the field resulting from a superposition of charges along the line segment $(-ia,ia)$, situated on the imaginary z-axis.

Although the explicit form of $f(t)$ is as yet unknown, we can, nevertheless, verify that ϕ_1 satisfies all the conditions listed in (2.2) except (iv), the aperture condition. That ϕ_1 satisfies property (i) follows directly from the fact that the functions $\exp\left[ik\, R^{\pm}(\rho,z;t)\right] / R^{\pm}(\rho,z;t)$ satisfy the wave equation in the region $z > 0$. To verify that ϕ_1 vanishes on the screen (property (ii)), we need only let z approach zero in equation (3.2) and make use of the specified behavior of the square roots [see equation (3.4)]. To verify property (iii), we first introduce the polar coordinates

$$(3.6) \qquad \begin{aligned} z &= R\cos\theta \ , \\ \rho &= R\sin\theta \ , \qquad 0 \leqq \theta \leqq \pi/2 \ , \\ R^2 &= \rho^2 + z^2 \ , \end{aligned}$$

and then note that the behavior of $R^{\pm}(\rho,z;t)$ for large R is:

$$(3.7) \qquad R^{\pm}(\rho,z;t) = R \pm iat\cos\theta + O\left(\frac{1}{R}\right) \ .$$

From this relation, it follows immediately that the Sommerfeld radiation condition is satisfied - specifically, that

$$\lim_{R\to\infty} \left\{ R\left(\frac{\partial\phi_1}{\partial R} - ik\,\phi_1 \right) \right\} = 0 \ , \quad 0 \leqq \theta \leqq \pi/2 \ .$$

Since $\exp[ikR^{\pm}(\rho,z;t)]/R^{\pm}(\rho,z;t)$ and its derivatives with respect to ρ and z are continuous functions of t in the half-space $z > 0$, we conclude that ϕ_1 and $\nabla\phi_1$ are finite in this region (see properties (v) and (vi)). To determine the behavior of ϕ_1 and $\nabla\phi_1$ near the edge, it is clearly sufficient, in view of the axial symmetry, to fix on a given cross section. Let T_δ be a torus whose axis of symmetry is the z-axis and whose equation in the cross section determined by the z-axis and the point (ρ,z) is (see Figure 2)

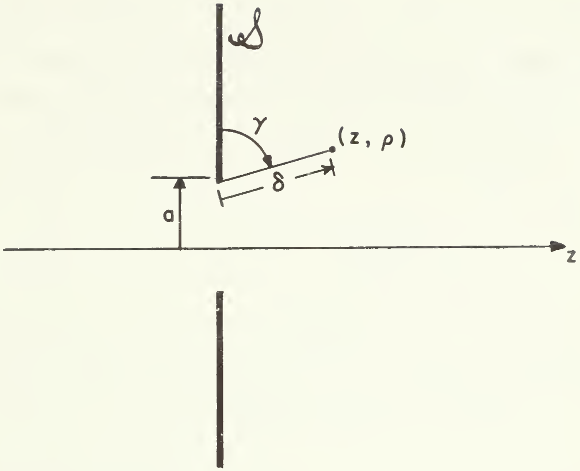

FIGURE 2

A cross-sectional view of the diffracting screen. The angle γ is measured from the screen clockwise to the ray joining the edge of the screen to the point (z,ρ); δ is the distance from the edge to the point (z,ρ).

$$(3.8) \qquad T_\delta : \quad \begin{cases} z = \delta \sin \gamma \,, \\ \rho = a + \delta \cos \gamma \,, \qquad 0 \leqq \gamma \leqq \pi \,. \end{cases}$$

Here, δ is the radius of the generating circle of T_δ and γ is the angle measured clockwise from the screen to the ray passing through the edge and the point (ρ,z).

Let us write

$$(3.9) \qquad \hat{\phi}_1(\delta,\gamma) = \phi_1(a + \delta \cos \gamma, \ \delta \sin \gamma).$$

Then, it is shown in Appendix I that*

$$(3.10) \qquad \hat{\phi}_1(\delta,\gamma) = \frac{-if_1(1)2\sqrt{2}}{a} (\delta/a)^{\frac{1}{2}} \sin (\gamma/2) + O(\delta/a) ,$$

and

$$(3.11) \qquad (\delta/a)^{\frac{1}{2}} \frac{\partial\hat{\phi}_1(\delta,\gamma)}{\partial\delta} = \frac{-i\sqrt{2}\, f_1(1)}{a^2} \sin (\gamma/2) + O\left[(\delta/a)^{\frac{1}{2}}\right] ,$$

$$(3.12) \qquad (\delta/a)^{\frac{1}{2}} \frac{1}{\delta} \frac{\partial\hat{\phi}_1(\delta,\gamma)}{\partial\gamma} = \frac{-i\sqrt{2}\, f_1(1)}{a^2} \cos (\gamma/2) + O\left[(\delta/a)^{\frac{1}{2}}\right] .$$

These equations and our earlier remarks show that ϕ_1 and $\nabla\phi_1$ satisfy the conditions (v) and (vi).

Note that ϕ_1 and $\nabla\phi_1$ in the neighborhood of the edge have the same form as the corresponding field quantities in the neighborhood of a perfectly soft half-plane.

Letting z approach zero in equation (3.2) and using equations (3.4) and (3.5), we find that the aperture field reduces to

$$(3.13) \qquad \phi_1(\rho,0) = \frac{2}{ai} \int_{\rho/a}^{1} \frac{\cosh\left[\alpha\sqrt{t^2-(\rho/a)^2}\right]}{\sqrt{t^2 - (\rho/a)^2}} f_1(t)dt .$$

---

* For sufficiently small values of α, $f_1(1) = ia\, a_1/2$, where a_1 is a power series in α, the first several terms of which are given in equation (4.27).

Moreover, setting

$$A_1(\theta) = -2\int_0^1 \sinh(\alpha t\cos\theta)\, f_1(t)dt \ , \tag{3.14}$$

we readily verify that the far field of ϕ_1 and the transmission coefficient t_1 can be expressed as follows (cf. reference [1], p.71):

$$\phi_1 \sim A_1(\theta)\ \frac{e^{ik\,R}}{R} \ , \tag{3.15}$$

$$t_1 = \frac{2}{a^2}\int_0^{\pi/2} |A_1(\theta)|^2 \sin\theta\, d\theta \ . \tag{3.16}$$

Approximate expressions for these quantities associated with normally-incident, plane-wave excitation are given in Sections 4 and 5.

4. Derivation of the Integral Equation for $f_1(t)$

The unknown function $f_1(t)$ may be determined by enforcing the aperture condition (2.3)-(iv), which, by virtue of equation (2.10), may be expressed as follows:

$$\lim_{z\downarrow 0} \frac{\partial\phi_1(\rho,z)}{\partial z} = \frac{\partial u^o(\rho,0)}{\partial z} \equiv u_1^o(\rho) \ , \quad 0 \leqq \rho < 1 \ . \tag{4.1}$$

We begin by calculating the limiting value of $\partial\phi_1/\partial z$. Assuming that $z > 0$, we differentiate both sides of equation (3.2) , make use of the identity

$$\frac{\partial M\left[R^{\pm}(\rho,z;t)\right]}{\partial z} = (z \pm iat)\ \frac{1}{\rho}\ \frac{\partial}{\partial\rho}\ M\left[R^{\pm}(\rho,z;t)\right] \ , \tag{4.2}$$

[M(w) is any analytic function of w] and obtain the following relation :

$$(4.3)\quad \frac{\partial\phi_1}{\partial z} = \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\int_0^1 \frac{(z + iat)\exp[ik\,R^+(\rho,z;t)]}{R^+(\rho,z;t)}\,f_1(t)dt$$

$$- \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\int_0^1 \frac{(z - iat)\exp[ik\,R^-(\rho,z;t)]}{R^-(\rho,z;t)}\,f_1(t)dt\ .$$

If we now let z approach zero and make use of the limiting behavior of $R^{\pm}(\rho,z;t)$ when $0 < \rho^2 \leq a^2t^2$ and $a^2t^2 \leq \rho^2 \leq a^2$, $0 \leq t \leq 1$, [see equations (3.4) and (3.5)], we find that for all ρ, $0 < \rho < a$,

$$(4.4)\quad \lim_{z\downarrow o}\left[\frac{\partial\phi_1(\rho,z)}{\partial z}\right] = \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\int_0^{\rho/a}\left\{\frac{\exp\left[i\alpha\sqrt{(\rho/a)^2-t^2}\right]}{\sqrt{(\rho/a)^2-t^2}} + \frac{\exp\left[i\alpha\sqrt{(\rho/a)^2-t^2}\right]}{\sqrt{(\rho/a)^2-t^2}}\right\} tf_1(t)dt$$

$$+ \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\int_{\rho/a}^1\left\{\frac{\exp\left[-\alpha\sqrt{t^2-(\rho/a)^2}\right]}{\sqrt{t^2-(\rho/a)^2}} - \frac{\exp\left[\alpha\sqrt{t^2-(\rho/a)^2}\right]}{\sqrt{t^2-(\rho/a)^2}}\right\} tf_1(t)dt$$

$$= \frac{2i}{\rho}\,\frac{\partial}{\partial\rho}\int_0^{\rho/a}\frac{\cos(\alpha\sqrt{(\rho/a)^2-t^2})}{\sqrt{(\rho/a)^2-t^2}}\,tf_1(t)dt$$

$$- \frac{2}{\rho}\,\frac{\partial}{\partial\rho}\int_0^{\rho/a}\frac{\sin(\alpha\sqrt{(\rho/a)^2-t^2})}{\sqrt{(\rho/a)^2-t^2}}\,tf_1(t)dt$$

$$- \frac{2}{\rho}\,\frac{\partial}{\partial\rho}\int_{\rho/a}^1\frac{\sinh(\alpha\sqrt{t^2-(\rho/a)^2})}{\sqrt{t^2-(\rho/a)^2}}\,tf_1(t)dt.$$

Since $\sin(\alpha\sqrt{(\rho/a)^2-t^2})\Big/\sqrt{(\rho/a)^2-t^2}$, $t^2 \geqq (\rho/a)^2$, is the continuation of $\sinh(\alpha\sqrt{(\rho/a)^2-t^2})\Big/\sqrt{(\rho/a)^2-t^2}$, $(\rho/a)^2 < t^2$, we conclude that for all ρ, $0 < \rho < a$,

$$(4.5)\quad \lim_{z\downarrow 0}\left[\frac{\partial\phi_1(\rho,z)}{\partial z}\right] = \frac{2i}{\rho}\frac{\partial}{\partial\rho}\int_0^{\rho/a}\frac{\cos(\alpha\sqrt{(\rho/a)^2-t^2})}{\sqrt{(\rho/a)^2-t^2}}\, tf_1(t)dt$$

$$-\frac{2}{\rho}\frac{\partial}{\partial\rho}\int_0^1\frac{\sinh(\alpha\sqrt{t^2-(\rho/a)^2})}{\sqrt{t^2-(\rho/a)^2}}\, tf_1(t)dt\ ,$$

where it must be understood that

$$(4.6)\quad \sqrt{t^2-(\rho/a)^2} = -\,i\sqrt{(\rho/a)^2-t^2}\quad \text{when } t^2 < (\rho/a)^2 .$$

If the value at $\rho = 0$ of the right member of equation (4.5) is defined to be the limit approached as ρ approaches zero, then it can be shown that (4.5) is valid for all ρ in the half-open interval $0 \leqq \rho < a$.

Invoking the aperture condition (4.1), we find that

$$(4.7)\quad \frac{\partial}{\partial\rho}\int_0^{\rho/a}\frac{\cos(\alpha\sqrt{(\rho/a)^2-t^2})}{\sqrt{(\rho/a)^2-t^2}}\, tf_1(t)dt$$

$$= \frac{\rho u_1^o(\rho)}{2i} + \frac{1}{i}\frac{\partial}{\partial\rho}\int_0^1\frac{\sinh\left[\alpha\sqrt{t^2-(\rho/a)^2}\right]}{\sqrt{t^2-(\rho/a)^2}}\, tf_1(t)dt\ ,$$

$$0 \leqq \rho < a .$$

Integrating both sides from 0 to ρ, we obtain the integral equation

$$(4.8)\qquad \int_0^{\rho/a} \frac{\cos\left(\alpha\sqrt{(\rho/a)^2-t^2}\right)}{\sqrt{(\rho/a)^2-t^2}}\, tf_1(t)dt = \frac{1}{2i}\int_0^{\rho} \zeta u_1^o(\zeta)d\zeta$$

$$+\frac{1}{i}\int_0^1 \frac{\sinh\left(\alpha\sqrt{t^2-(\rho/a)^2}\right)}{\sqrt{t^2-(\rho/a)^2}}\, tf_1(t)dt - C\ ,$$

where

$$(4.9)\qquad C = \frac{1}{i}\int_0^1 \frac{\sinh \alpha s}{s}\left[sf_1(s)\right] ds\ .$$

When we introduce the new variables

$$(4.10)\qquad \xi = t^2\ ,$$

$$(4.11)\qquad \eta = (\rho/a)^2\ ,$$

into both sides of (4.8), this equation assumes the following form:

$$(4.12)\qquad \int_0^{\eta} \frac{\cos\left(\alpha\sqrt{\eta-\xi}\right)}{\sqrt{\eta-\xi}}\, F(\xi)d\xi = G(\eta) + H(\eta)\ ,$$

where

$$(4.13)\qquad F(\xi) = f(\sqrt{\xi}\,)/2\ ,$$

$$(4.14)\qquad G(\eta) = \frac{1}{2i}\int_0^{a\sqrt{\eta}} \zeta u_1^o(\zeta)d\zeta\ ,$$

$$(4.15)\qquad H(\eta) = \frac{1}{i}\int_0^1 \frac{\sinh\left(\alpha\sqrt{\xi-\eta}\right)}{\sqrt{\xi-\eta}}F(\xi)d\xi - C\ .$$

Using the fact that the Laplace transform of a convolution of two functions is the product of the Laplace transforms, we may solve equation (4.12) for $F(\xi)$. The final result is

$$(4.16)\quad F(\eta) = \frac{1}{\pi}\frac{d}{d\eta}\int_0^\eta \frac{\cosh \alpha\sqrt{\eta-\mu}}{\sqrt{\eta-\mu}}\left[G(\mu) + H(\mu)\right] d\mu$$

$$= g(\eta) + h(\eta) \ ,$$

where $g(\eta)$ and $h(\eta)$ are defined by

$$(4.17)\quad g(\eta) = \frac{1}{\pi}\frac{d}{d\eta}\int_0^\eta \frac{\cosh(\alpha\sqrt{\eta-\xi})}{\sqrt{\eta-\xi}} G(\xi)d\xi = \frac{1}{2\pi i}\frac{d}{d\eta}\int_0^\eta \frac{\cosh(\alpha\sqrt{\eta-\xi})}{\sqrt{\eta-\xi}}$$

$$\times\left(\int_0^{a\sqrt{\xi}} \zeta u_1^0(\zeta)d\zeta\right)d\xi \ ,$$

$$(4.18)\quad h(\eta) = \frac{1}{\pi}\frac{d}{d\eta}\int_0^\eta \frac{\cosh(\alpha\sqrt{\eta-\mu})}{\sqrt{\eta-\mu}} H(\mu)d\mu$$

$$= \frac{1}{\pi i}\int_0^1 F(\xi)\left[\frac{d}{d\eta}\int_0^\eta \frac{\cosh(\alpha\sqrt{\eta-\mu})\ \sinh(\alpha\sqrt{\eta-\mu})}{\sqrt{\eta-\mu}\ \sqrt{\xi-\mu}}\, d\mu\right]d\xi$$

$$-\ \frac{C}{\pi}\ \cosh \alpha\sqrt{\eta}$$

The expression for $g(\eta)$ may be simplified somewhat, first by integrating by parts and then making the change of variables $\xi = \eta - \eta s^2$.

The result is:

$$(4.19)\quad g(\eta) = \frac{a^2\sqrt{\eta}}{2\pi i}\int_0^1 \cosh\,(\alpha\sqrt{\eta}\,s)u_1^o\left[a\sqrt{\eta(1-s^2)}\right]ds$$

The expression for $h(\eta)$ may also be simplified* - by (1), making use of the addition formula for hyperbolic functions, (2) introducing obvious changes of variables, (3) carrying out the differentiation with respect to η. The final result is:

$$(4.20)\quad h(\eta) = \frac{1}{2\pi i}\int_0^1 F(\xi)\left\{\frac{\sinh\left[\alpha(\sqrt{\xi}-\sqrt{\eta})\right]}{\sqrt{\eta}\,(\sqrt{\xi}-\sqrt{\eta})} + \frac{\sinh\left[\alpha(\sqrt{\xi}+\sqrt{\eta}\right]}{\sqrt{\eta}\,(\sqrt{\xi}+\sqrt{\eta})}\right\}d\xi$$

$$- C\cosh\,(\alpha\sqrt{\eta})/\pi\ .$$

Combining equations (4.16)-(4.20), we find:

$$(4.21)\quad F(\eta) = \frac{a^2\sqrt{\eta}}{2\pi i}\int_0^1 \cosh\,(\alpha\sqrt{\eta}\,s)u_1^o\left[a\sqrt{\eta\,(1-s^2)}\right]ds$$

$$+ \frac{1}{2\pi i}\int_0^1 F(\xi)\left\{\frac{\sinh\alpha(\sqrt{\xi}-\sqrt{\eta})}{\sqrt{\eta}\,(\sqrt{\xi}-\sqrt{\eta})} + \frac{\sinh\alpha(\sqrt{\xi}+\sqrt{\eta}}{\sqrt{\eta}\,(\sqrt{\xi}+\sqrt{\eta})}\right\}d\xi$$

$$- C\cosh\left(\alpha\sqrt{\eta}\right)/\,\pi\ .$$

Replacing η by t^2 and ξ by s^2 wherever ξ and η appear in (4.21) and taking into account (1) the definitions of C [equation (4.9)] and $F(\xi)$ [equation (4.13)] and (2) the fact that $f_1(t)$ is an odd function of t, we obtain the following integral equation for the function $tf_1(t)$:

---

* Cf. reference [6] pp. 10-11.

$$(4.22)\quad tf_1(t) = \frac{a^2t^2}{\pi i}\int_0^1 \cosh(\alpha ts)v_1^{\circ}\left[at\sqrt{(1-s^2)}\,\right]ds$$

$$+\frac{1}{\pi i}\int_{-1}^{1}\left\{\frac{\sinh[\alpha(t-s)]}{t-s}-\frac{\cosh(\alpha t)\sinh(\alpha s)}{s}\right\} sf_1(s)ds\ .$$

In the special case of normal plane-wave incidence, where $u_1^{\circ}(\rho)=ik$ [see equations (2.10) and (4.1)], equation (4.22) becomes

$$(4.23)\quad tf_1(t) = \frac{at}{\pi}\sinh\alpha t + \frac{1}{\pi i}\int_{-1}^{1}\left\{\frac{\sinh[\alpha(t-s)]}{t-s}-\frac{\cosh(\alpha t)\sinh(\alpha s)}{s}\right\} sf_1(s)ds,$$

It is easily verified that the kernel in the integral equation (4.22) is of order α^3 when α is small. Thus, assuming merely that $u_1^{\circ}(\rho)$, $\rho > 0$, is a continuous function of ρ, we readily conclude that equation (4.22) has a unique continuous solution (the Neumann series) for sufficiently small values of α[**]. With this fact established, it readily follows that the domain of regularity of $tf_1(t)$ contains the circular domain $\mathcal{C}_\Delta$ defined by the relation $0 \leq |t| \leq 1 + \Delta/a$. To prove this, it is enough to observe that (1) the integral term is an entire function of t since the kernel enjoys this property and (2) the inhomogeneous term is regular for all t in the domain $\mathcal{C}_\Delta$ because $u_1^{\circ}(\rho)$ is, by hypothesis, a regular function of ρ in the circular domain $0 \leq \rho \leq a + \Delta$.

---

[*] This justifies our earlier use of Laplace transform methods.

[**] In problem $\mathcal{P}_2$ we are able to establish the unique existence of $f_2(t)$ for all $\alpha > 0$ because the kernel $\sinh[\alpha(t-s)]/t-s$ (see equations (5.11) and (5.13)) is symmetric and because the complex constant $\lambda = 1/\pi i$ can never be an eigenvalue of the corresponding homogeneous integral equation.

In effect, the regularity properties of $u_1^o(\rho)$ control the regularity properties of $tf_1(t)$. That $tf_1(t)$ is even follows from the form of the kernel and the fact that the inhomogeneous term is even - it is here that we make use of the evenness of $u_1^o(\rho)$ as a function of ρ. Finally, let us note that $f_1(t) = \mathcal{O}(e^{\alpha|t|})$ when $|t|$ becomes large so that its Laplace transform exists*.

Assuming ϕ_1 exists, it remains to prove that ϕ_1 is unique in the class of solutions satisfying the properties listed in equation (2.3). To this end, we employ the standard argument which, briefly, runs as follows. If ϕ_1' is another solution, then $v = \phi_1 - \phi_1'$ enjoys all the properties of ϕ_1 except that $\partial v/\partial z$ vanishes in the aperture. We next consider the integral $\int_{\mathcal{D}} \nabla(v^* \cdot \nabla v)dV$; here, v^* denotes the complex conjugate of v, $\mathcal{D}$ is the domain contained within the surface shown (in cross section) in Figure 3 and dV is the volume element.

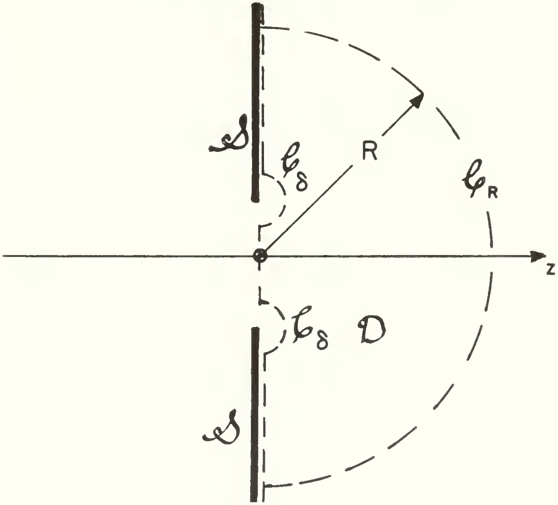

FIGURE 3

A cross-sectional view of the surfaces (shown dashed) that bound the domain $\mathcal{D}$. $\mathcal{C}_R$ and $\mathcal{C}_\delta$ are meridianal semi-circles cut out of a hemisphere and torus whose equatorial planes coincide with the plane of the screen $\mathcal{S}$. Both torus and hemisphere are centered at 0.

Employing (1) Green's Theorem, (2) the vanishing of v and $\partial v/\partial z$ on the aperture and the screen, respectively, and (3) the fact that v satisfies the wave equation and the radiation condition, we find, on letting ϵ and R^{-1} approach zero, that $\int_{\mathcal{D}} (|\nabla v|^2 + k^2 |v|^2) dV = 0$. From this, it follows easily that v vanishes everywhere - i.e., that $\phi_1 = \phi_1'$.

Turning now to the special case of normal, plane-wave incidence, we use the fact that the kernel is of the order α^3 when α is small, and calculate several terms of the Neumann series for $tf_1(t)$. We find, after three iterations, that $f_1(t)$[*] is:

$$
\begin{aligned}
(4.24)\quad f_1(t) = \frac{a}{\pi} \Bigg\{ \Bigg[& \alpha t + \frac{\alpha^3 t^3}{6} + \frac{\alpha^5 t^5}{120} + \frac{\alpha^7 t^7}{5040} + \frac{\alpha^9 t^9}{362880} \\
& - \frac{4}{\pi^2} \left(\left[\frac{\alpha^7}{81} + \frac{7\alpha^9}{2025} \right] t + \frac{\alpha^9}{810} t^3 \right) \Bigg] \\
& + \frac{2i\alpha}{\pi} \Bigg[\Bigg(\left[\frac{\alpha^3}{9} + \frac{4\alpha^5}{225} + \frac{\alpha^7}{735} + \frac{8\alpha^9}{127575} \right] t \\
& + \left[\frac{\alpha^5}{90} + \frac{\alpha^7}{525} + \frac{31\alpha^9}{198450} \right] t^3 + \left[\frac{\alpha^7}{2520} + \frac{\alpha^9}{14175} \right] t^5 + \frac{\alpha^9}{136080} t^7 \Bigg) \\
& - \frac{4}{\pi^2} \left(\frac{\alpha^9}{729} \right) t \Bigg] \Bigg\} + \mathcal{O}\left(\alpha^{11}\right) .
\end{aligned}
$$

---

[*] The maximum value of α for which this expression for $f_1(t)$, regarded as a power series in α, converges is at present unknown.

Inserting this expression for $f_1(t)$ into equations (3.13), (3.14) and (3.16), we find that the aperture field is

$$(4.25)\quad \phi_1 = \sum_{n=1}^{\infty} a_n\left[1 - (\rho^2/a^2)\right]^{n-1/2} ,$$

where

$$(4.26)\quad a_1 = \frac{2\alpha}{\pi i}\left[1 + \frac{\alpha^2}{6} + \frac{2i\alpha^3}{9\pi} + \frac{\alpha^4}{120} + \frac{13i\alpha^5}{225\pi} + \left(\frac{1}{5040} - \frac{4}{81\pi^2}\right)\alpha^6\right.$$

$$+ \frac{323i\alpha^7}{44100\pi} + \left(\frac{1}{362880} - \frac{38}{2025\pi^2}\right)\alpha^8$$

$$\left. + \left(\frac{269i}{5715360\pi} - \frac{4i}{729\pi^3}\right)\alpha^9\right] + \mathcal{O}\left(\alpha^{11}\right),$$

$$a_2 = \frac{\alpha^3}{a\pi i}\left[1 + \frac{3\alpha^2}{10} + \frac{2i\alpha^3}{5\pi} + \frac{\alpha^4}{56} + \frac{19i\alpha^5}{175\pi} + \left(\frac{1}{2160} - \frac{4}{45\pi^2}\right)\alpha^6\right.$$

$$\left. + \frac{631\alpha^7}{396900}\right] + \mathcal{O}\left(\alpha^{11}\right),$$

$$a_3 = \frac{\alpha^5}{300\pi i}\left[1 + \frac{5\alpha^2}{14} + \frac{10i\alpha^3}{21\pi} + \frac{5\alpha^4}{216} + \frac{76i\alpha^5}{567\pi}\right] + \mathcal{O}\left(\alpha^{11}\right),$$

$$a_4 = \frac{\alpha^7}{17640\pi i}\left[1 + \frac{7\alpha^2}{18} + \frac{58\alpha^3 i}{81\pi}\right] + \mathcal{O}(\alpha^{11}) \; ,$$

$$a_5 = \frac{\alpha^9}{(1632960)\pi i} + \mathcal{O}(\alpha^{11}) \; .$$

The diffracted amplitude is:

$$(4.27) \quad A_1(\theta) = \frac{-2a}{\pi}\left\{\left[\left(\frac{\alpha^2}{3} + \frac{\alpha^4}{30} + \frac{\alpha^6}{840} + \frac{\alpha^8}{45360} + \frac{\alpha^{10}}{3991680}\right)\cos\theta\right.\right.$$

$$+ \left(\frac{\alpha^4}{30} + \frac{\alpha^6}{252} + \frac{\alpha^8}{6480} + \frac{\alpha^{10}}{332640}\right)\cos^3\theta + \left(\frac{\alpha^6}{840} + \frac{\alpha^8}{6480} + \frac{\alpha^{10}}{158400}\right)\cos^5\theta$$

$$\left. + \left(\frac{\alpha^8}{45360} + \frac{\alpha^{10}}{332640}\right)\cos^7\theta + \left(\frac{\alpha^{10}}{3991680}\right)\cos^9\theta\right]$$

$$- \frac{4}{\pi^2}\left[\left(\frac{\alpha^8}{243} + \frac{17\alpha^{10}}{12150}\right)\cos\theta + \frac{\alpha^{10}}{2430}\cos^3\theta\right] + \frac{2i\alpha}{\pi}\left[\left(\frac{\alpha^4}{27} + \frac{11\alpha^6}{1350}\right.\right.$$

$$\left.\left.\left. + \frac{131\alpha^8}{147000}\right)\cos\theta + \left(\frac{\alpha^6}{270} + \frac{3\alpha^8}{3500}\right)\cos^3\theta + \frac{\alpha^8}{7560}\cos^5\theta\right]\right\} + \mathcal{O}(\alpha^{11}) \; .$$

The transmission coefficient is:

$$(4.28) \quad t_1 = \frac{8\alpha^4}{27\pi^2}\left[1 + \frac{8\alpha^2}{25} + \frac{311\alpha^4}{6125} + \left(\frac{2612}{496125} - \frac{4}{81\pi^2}\right)\alpha^6\right] + \mathcal{O}(\alpha^{12}).$$

These expressions are in complete agreement with those given by C.J. Bouwkamp (see reference [1] pp. 64 and 72). Actually, our result for the aperture field is a little more accurate than that given by Bouwkamp; we have calculated $a_1, a_2, \ldots, a_5$, regarded as power series in α, up to and including α^{10}; Bouwkamp gives $a_1, \ldots, a_4$ up to and including α^8.

This completes our discussion of the first boundary-value problem.

5. The Second Boundary-Value Problem

We represent $\phi_2(\rho, z)$ as follows:

$$(5.1) \quad \phi_2(\rho,z) = \int_{-1}^{1} \frac{e^{ik\,R^+(\rho,z;t)}}{R^+(\rho,z;t)} f_2(t)dt$$

$$= \int_{0}^{1} \left\{ \frac{e^{ik\,R^+(\rho,z;t)}}{R^+(\rho,z;t)} + \frac{e^{ik\,R^-(\rho,z;t)}}{R^-(\rho,z;t)} \right\} f_2(t)dt \; .$$

The quantities $R^{\pm}(\rho,z;t)$ are defined in equation (3.3); $f_2(t)$ is required to be (1) an even function of t and (2) a regular function of t in a circular domain of the complex t plane containing the unit disc. At points of the aperture, the only points in physical space where the integrand can be singular, ϕ_2 and its derivatives may be defined as the limits approached by ϕ_2 in $z > 0$ as z approaches zero*.

---

* In the expression for ϕ_1 [equation (3.1)], the pole singularity of the integrand associated with $\rho = z = 0$ is only apparent since $f_1(t)$ is odd. On the other hand, the integrand of ϕ_2 at $\rho = z = 0$ has a genuine pole. In this case, $\lim_{z \downarrow 0} \phi(0,z)$ may, for example, be obtained from the well-known Plemelj formulas (see reference [11]).

Repeating the argument of Section 3, we find that conditions (i), (ii)$'$ and (iii)(see Section 2) are satisfied. In Appendix I, we show that ϕ_2 and $\nabla\phi_2$ have the following behavior in the neighborhood of the edge*:

$$(5.2)\qquad \widehat{\phi}_2 = 1 - \frac{2\sqrt{2}\, f_2(1)}{a}\ (\delta/a)^{\frac{1}{2}} \cos(\gamma/2) + \mathcal{O}(\delta/a) \quad ,$$

$$(5.3)\qquad (\delta/a)^{\frac{1}{2}} \frac{\partial\widehat{\phi}_2}{\partial\delta} = \frac{-\sqrt{2}\, f_2(1)}{a^2}\ \cos\,(\gamma/2)\ + \mathcal{O}\left[(\delta/a)^{\frac{1}{2}}\right] \quad ,$$

$$(5.4)\qquad (\delta/a)^{\frac{1}{2}}\ \frac{1}{\delta}\ \frac{\partial\widehat{\phi}_2}{\partial\gamma} = \frac{\sqrt{2}\, f_2(1)}{a^2}\ \sin\,(\gamma/2) + \mathcal{O}\left[(\delta/a)^{\frac{1}{2}}\right] \ .$$

The variables δ and γ are those introduced earlier in equation (3.8) (see Figure 2) and $\widehat{\phi}_2 = \widehat{\phi}(\delta,\gamma)$ is defined by

$$(5.5)\qquad \widehat{\phi}_2 = \phi_2\ (\delta\cos\gamma,\ a + \delta\sin\gamma)\ .$$

These results imply that conditions(v) and (vi) of Section 2 are satisfied.

Employing equations (3.4), (3.5) and (3.7) , we obtain from (5.1) the following expressions for $\partial\phi_2/\partial z$ (in the aperture), the far field and the transmission coefficient t_2:

$$(5.6)\qquad \frac{\partial\phi_2}{\partial z} = \frac{2}{\rho}\frac{\partial}{\partial\rho}\int_{\rho/a}^{1} \frac{\cosh\left[\alpha\sqrt{t^2-(\rho/a)^2}\right]}{\sqrt{t^2-(\rho/a)^2}}\ t f_2(t)dt\ , \quad 0 \leq \rho < a\ ,$$

$$(5.7)\qquad \phi_2 \sim A_2(\theta)\ \frac{e^{ik\,R}}{R}\ , \quad R \rightarrow \infty\ ,$$

---

* $f_2(1) = \frac{a}{\pi}\, a_o$, where a_o is defined in equation (5.15)below. Several terms of the power series for a_o, regarded as a function of α, are given in equation (5.16).

where

$$(5.8)\qquad A_2(\theta) = 2\int_0^1 \cosh(\alpha t\cos\theta) f_2(t)\,dt;$$

$$(5.9)\qquad t_2 = \frac{2}{a^2}\int_0^{\pi/2} |A_2(\theta)|^2 \sin\theta\, d\theta\ .$$

Enforcing the aperture condition, we find the following analog of equation (4.8):

$$(5.10)\qquad \int_0^{\rho/a} \frac{\cos\left[\alpha\sqrt{(\rho/a)^2-t^2}\right]}{\sqrt{(\rho/a)^2-t^2}} f_2(t)\,dt = \frac{a}{2}u_2^{o}(\rho) + \frac{1}{i}\int_0^1 \frac{\sinh\left(\alpha\sqrt{t^2-(\rho/a)^2}\right)}{\sqrt{t^2-(\rho/a)^2}} \times f_2(t)\,dt\ .$$

Arguing as in Section 4, we are led to the integral equation

$$(5.11)\qquad f_2(t) = g(t) + \frac{1}{\pi i}\int_{-1}^1 \sinh\frac{[\alpha(t-s)]}{t-s} f_2(s)\,ds,$$

where

$$(5.12)\qquad g(t) = \frac{a}{\pi}u_2^{o}(0)\cosh\alpha t + \frac{a^2 t}{\pi}\cdot\int_0^1 \cosh\alpha t\,\frac{u_2^{o\prime}(at\sqrt{1-s^2})\,ds}{\sqrt{1-s^2}}$$

$\left(u_2^{o\prime}(\rho) = \frac{d}{d\rho}u_2^{o}(\rho)\right)$. In the special case of normally incident, plane-wave excitation of unit amplitude, we find, since $u_2^{o\prime}$ vanishes, that

$$(5.13)\qquad f_2(t) = \frac{a}{\pi}\cosh\alpha t + \frac{1}{\pi i}\int_{-1}^1 \sinh\frac{[\alpha(t-s)]}{(t-s)} f_2(s)\,ds\ .$$

Equation (5.11) implies the unique existence of an even function of t, regular in the domain $|t| \leqq 1 + \Delta/a$, if, as we have assumed, $u_2^0(\rho)$ is an even function of ρ, regular in the domain $0 \leqq |\rho| \leqq a + \Delta$. Arguments paralleling those employed in Section 4 guarantee the existence of the Laplace transform of $f_2(\sqrt{\xi})/\sqrt{\xi}$, so that the manipulation leading to (5.11)-(5.13) is justified. Making use of the fact that the kernel in (5.11) or (5.13) is of order α when α is small, we may obtain f as a Neumann series. In the plane-wave case [equation (5.13)], we find that *

$$
\begin{aligned}
(5.14) \qquad f_2(t) = \frac{a}{\pi} \Bigg\{ & \left[1 + \frac{\alpha^2 t^2}{2} + \frac{\alpha^4 t^4}{24} + \frac{\alpha^6 t^6}{720}\right] \\
& - \frac{1}{\pi}\left[\left(2\alpha + \frac{4\alpha^3}{9} + \frac{4\alpha^5}{75}\right) + \left(\frac{\alpha^3}{3} + \frac{4\alpha^5}{45}\right)t^2 + \frac{\alpha^5}{60}t^4\right] \\
& - \frac{1}{\pi^2}\left[\left(4\alpha^2 + \frac{4\alpha^4}{3} + \frac{508\alpha^6}{2025}\right) + \left(\frac{2\alpha^4}{3} + \frac{34\alpha^6}{135}\right)t^2 + \frac{\alpha^6}{30}t^4\right] \\
& + \frac{1}{\pi^3}\left[\left(8\alpha^3 + \frac{32\alpha^5}{9}\right) + \frac{4\alpha^5}{3}t^2\right] + \frac{1}{\pi^4}\left[\left(16\alpha^4 + \frac{80\alpha^6}{9}\right) + \frac{8\alpha^6}{3}t^2\right] \\
& - \frac{1}{\pi^5}\left[32\alpha^5\right] - \frac{1}{\pi^6}\left[64\alpha^6\right]\Bigg\} + O(\alpha^7) .
\end{aligned}
$$

---

* The maximum radius of convergence for this sum regarded as a function of α is not known. C.H. Yang [12] has shown that it is at least 1/2.

Inserting this expression into equations (5.6), (5.8) and (5.9), we obtain the following expressions for $\partial\phi_2/\partial z$, $A_2(\theta)$ and t_2:

$$(5.15) \qquad \frac{\partial\phi_2}{\partial z} = -\frac{2}{a\pi}\Big[a_o[1-(\rho/a)^2]^{-\frac{1}{2}} + a_1[1-(\rho/a)^2]^{\frac{1}{2}} + a_2[1-(\rho/a)^2]^{\frac{3}{2}} + a_3[1-(\rho/a)^2]^{\frac{5}{2}}\Big] + \mathcal{O}(\alpha^7) ,$$

where

$$(5.16) \qquad a_o = 1 - \frac{2i}{\pi}\alpha + \left(\frac{1}{2} - \frac{4}{\pi^2}\right)\alpha^2 - i\left(\frac{7}{9\pi} - \frac{8}{\pi^3}\right)\alpha^3 + \left(\frac{1}{24} - \frac{2}{\pi^2} + \frac{16}{\pi^4}\right)\alpha^4 - i\left(\frac{143}{900\pi} - \frac{44}{9\pi^3} + \frac{32}{\pi^5}\right)\alpha^5 + \left(\frac{1}{720} - \frac{2171}{4050\pi^2} + \frac{104}{9\pi^4} - \frac{64}{\pi^6}\right)\alpha^6 + \mathcal{O}(\alpha^7) ;$$

$$(5.17) \qquad a_1 = -\frac{1}{2}\alpha^2 - \frac{i\alpha^3}{3\pi} + \left(\frac{1}{12} - \frac{2}{3\pi^2}\right)\alpha^4 - i\left(\frac{13}{90\pi} - \frac{4}{3\pi^3}\right)\alpha^5 + \left(\frac{1}{80} - \frac{49}{135\pi^2} + \frac{8}{3\pi^4}\right)\alpha^6 + \mathcal{O}(\alpha^7) ;$$

$$(5.18) \qquad a_2 = -\frac{1}{72}\alpha^4 - i\frac{\alpha^5}{60\pi} + \left(\frac{1}{240} - \frac{1}{30\pi^2}\right)\alpha^6 + \mathcal{O}(\alpha^7) ;$$

$$(5.19) \qquad a_3 = -\frac{\alpha^6}{3600} + \mathcal{O}(\alpha^7) .$$

The diffracted amplitude is

$$
\begin{aligned}
(5.20) \qquad \frac{\pi}{2a} A_2(\theta) = \Bigg\{ & 1 + \left(\frac{1}{3} - \frac{4}{\pi^2} - \frac{1}{6} \sin^2\theta \right) \alpha^2 \\
& + \left[\frac{16}{\pi^4} - \frac{20}{9\pi^2} + \frac{1}{15} + \left(\frac{2}{3\pi^2} - \frac{1}{15} \right) \sin^2\theta + \frac{1}{120} \sin^4\theta \right] \alpha^4 \\
& + \left[\left(\frac{112}{9\pi^4} + \frac{2}{315} - \frac{64}{\pi^6} - \frac{448}{675\pi^2} \right) + \left(\frac{16}{45\pi^2} - \frac{8}{3\pi^4} - \frac{1}{105} \right) \sin^2\theta \right. \\
& \left. + \left(\frac{1}{280} - \frac{1}{30\pi^2} \right) \sin^4\theta - \frac{\sin^6\theta}{5040} \right] \alpha^6 \Bigg\} \\
& + \frac{2}{\pi i} \Bigg\{ \alpha + \left(- \frac{4}{\pi^2} + \frac{4}{9} - \frac{\sin^2\theta}{6} \right) \alpha^3 + \left[\left(\frac{16}{\pi^4} - \frac{8}{3\pi^2} + \frac{71}{675} \right) \right. \\
& \left. + \left(\frac{2}{3\pi^2} - \frac{19}{270} \right) \sin^2\theta + \frac{1}{120} \sin^4\theta \right] \alpha^5 \Bigg\} + \mathcal{O}(\alpha^6) .
\end{aligned}
$$

The transmission coefficient is

$$
\begin{aligned}
(5.21) \qquad t_2 = \frac{8}{\pi^2} \Bigg[& 1 + \left(\frac{4}{9} - \frac{4}{\pi^2} \right) \alpha^2 + \left(\frac{71}{675} - \frac{8}{3\pi^2} + \frac{16}{\pi^4} \right) \alpha^4 \\
& + \left(\frac{568}{33075} - \frac{1936}{2025\pi^2} + \frac{128}{9\pi^4} - \frac{64}{\pi^6} \right) \alpha^6 \Bigg] + \mathcal{O}(\alpha^8) .
\end{aligned}
$$

Our expression for t_2 agrees completely with that obtained by C.J. Bouwkamp[1] and, where comparison is possible, our expressions for $\partial\phi_2/\partial z$ and $A_2(\theta)$ agree with the listed results in reference [1] (see p. 71)* . Note, however, that our calculations for $\partial\phi_2/\partial z$ and $A_2(\theta)$ have both been carried out to terms of order α^6 whereas the corresponding results listed in reference [1] have been carried out only as far as α^4.

Acknowledgments

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* A misprint occurs in the expression for $A_2(\theta)$ on page 71. It may be rectified by replacing $A_2(\theta)$ by $A_2(\theta)/4$.

Appendix I

Behavior of the Fields near the Circular Edge

Let $I_f^{\pm} = I_f^{\pm}(\rho, z)$ denote the integrals

$$(I.1) \qquad I_f^{\pm} = \int_0^1 \frac{e^{ik\sqrt{\rho^2 + (z \pm iat)^2}}}{\sqrt{\rho^2 + (z \pm iat)^2}} f(t)dt \quad ,$$

where the upper and lower signs in the left member correspond, respectively, to the upper and lower signs in the right member. Then ϕ_1 and ϕ_2 may be expressed as follows:

$$(I.2) \qquad \phi_1 = I_{f_1}^{+} - I_{f_1}^{-} \quad ,$$

$$(I.3) \qquad \phi_2 = I_{f_2}^{+} - I_{f_2}^{-} \quad .$$

Let $f(t)$ be regular in a circular domain* $0 \leqq |t| \leqq 1 + \Delta/a$; let Δ' be defined as follows:

$$(I.4) \qquad \Delta'/a = \min\left\{\frac{1}{4}, \frac{\Delta}{a}\right\} > 0 \; .$$

We shall show that

$$(I.5) \quad I_f^{\pm} = \frac{1}{a}\int_0^1 \frac{\exp(i\alpha\sqrt{1-s^2})}{\sqrt{1-s^2}} f(s)ds - \frac{\sqrt{2}\, f(1)}{a}\left(\frac{\delta}{a}\right)^{\frac{1}{2}} e^{\pm i\gamma/2} + \mathcal{O}(\delta/a) \; ,$$

---

* Recall that $f_1(t)$ and $f_2(t)$ are regular in the circular domain $0 \leqq |t| \leqq 1 + \Delta/a, \Delta > 0$.

$$(I.6) \qquad \frac{\partial I_f^{\pm}}{\partial \delta} = - \frac{f(1)}{\sqrt{2}\, a^2} \left(\frac{\delta}{a} \right)^{-\frac{1}{2}} e^{\pm i\gamma/2} + \mathcal{O}(1) ,$$

$$(I.7) \qquad \frac{1}{\delta} \frac{\partial I_f^{\pm}}{\partial \gamma} = \frac{\mp i f(1)}{\sqrt{2}\, a^2} \left(\frac{\delta}{a} \right)^{-\frac{1}{2}} e^{\pm i\gamma/2} + \mathcal{O}(1) ,$$

for all δ/a such that

$$(I.8) \qquad 0 < \delta/a < \Delta'/a ;$$

the variables δ and γ are the coordinates introduced earlier in equation (3.8) (see Figure 2). Combining equations (I.5)-(I.7) and (I.2)-(I.3), we obtain the results quoted in the text - namely, those appearing in equations (3.10)-(3.12) and (5.2)-(5.4).

We shall derive (I.5)-(I.7) for I_f^+ only; the corresponding derivations for the I_f^--quantities are exactly analogous.

We begin with the observation that

$$(I.9) \qquad R^+ = \sqrt{\rho^2 + (z + iat)^2} = a \sqrt{\left(\frac{\rho + iz}{a} - t \right)} \sqrt{\left(\frac{\rho - iz}{a} + t \right)} .$$

Introducing the edge-coordinates δ and γ [see equation (3.8)], we find that R^+ may be expressed as follows:

$$(I.10) \qquad R^+ = a \sqrt{(1 - t) + (\delta/a)\, e^{i\gamma}} \cdot \sqrt{(1 + t) + (\delta/a)\, e^{-i\gamma}}$$

$$= a\, d_1\, d_2$$

Here, d_1 and d_2 represent the first and second square roots respectively.

The branches of d_1 and d_2 are determined by the requirement that their product behaves like R^+. Such a determination is implied by the formulas

$$\arg d_1 = \frac{1}{2}\tan^{-1}\left[\frac{(\delta/a)\sin\gamma}{(1-t)+(\delta/a)\cos\gamma}\right],$$

(I.11)

$$\arg d_2 = -\frac{1}{2}\tan^{-1}\left[\frac{(\delta/a)\sin\gamma}{(1+t)+(\delta/a)\cos\gamma}\right],$$

where the branch of $\tan^{-1}x$ is fixed by requiring that

$$0 \leqq \tan^{-1} x < \pi \;. \tag{I.12}$$

Combining (I.9) and(I.1), we find that

$$I_f^+ = \int_0^1 \frac{e^{i\alpha\left[\sqrt{1-t+(\delta/a)e^{i\gamma}}\;\sqrt{1+t+(\delta/a)e^{-i\gamma}}\right]}}{\sqrt{1-t+(\delta/a)e^{i\gamma}}\;\sqrt{1+t+(\delta/a)e^{-i\gamma}}}\, f(t)dt \;. \tag{I.13}$$

Mapping the complex t-plane into the complex ζ-plane by means of the transformation

$$\zeta = 1 - t + (\delta/a)\, e^{i\gamma}\,, \tag{I.14}$$

and setting

$$\xi = \frac{2(\delta/a)\cos\gamma}{2-\zeta}\,, \tag{I.15}$$

we may re-express I_f^+ as

$$I_f^+ = -\frac{1}{a}\int_P \frac{\exp(i\alpha\sqrt{\zeta}\,\sqrt{2-\zeta}\,\sqrt{1+\xi}\;)}{\sqrt{\zeta}\,\sqrt{2-\zeta}\,\sqrt{1+\xi}}\, f\left[1-\zeta+(\delta/a)e^{i\gamma}\right]d\zeta \;, \tag{I.16}$$

where the path P is given parametrically by:

$$\zeta(\mu) = 1 - \mu + (\delta/a)\, e^{i\gamma} \quad , \; 0 \leq \mu \leq 1. \tag{I.17}$$

(See Figure 4).

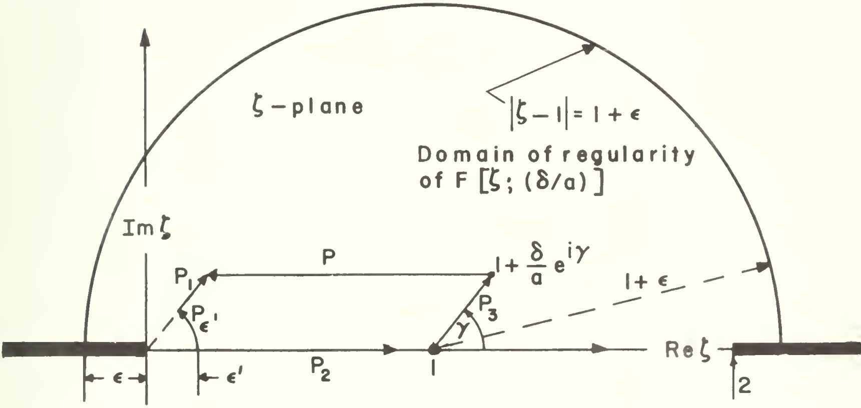

FIGURE 4

The ζ-plane.

I_f^+ may now be rewritten in the following more convenient form:

$$(I.18)\qquad I_f^+ = -\frac{1}{a}\int_P \frac{\exp(i\alpha\sqrt{\zeta}\sqrt{2-\zeta}\,)}{\sqrt{\zeta}\sqrt{2-\zeta}}\, f(1-\zeta)d\zeta + \left(\frac{\delta}{a}\right)\int_P \frac{G[\zeta;(\delta/a)]}{\sqrt{\zeta}\sqrt{2-\zeta}}\, d\zeta\ ,$$

where

$$(I.19)\qquad G[\zeta;(\delta/a)] = \frac{1}{(\delta/a)}\left[\frac{\exp\left[i\alpha\sqrt{\zeta}\sqrt{2-\zeta}\sqrt{1+\xi}\,\right]}{\sqrt{1+\xi}}\, f\left[1-\zeta+(\delta/a)e^{i\gamma}\right] - \exp\left[i\alpha\sqrt{\zeta}\sqrt{2-\zeta}\right]f(1-\zeta)\right].$$

Note that

$$(I.20)\qquad |\xi| = \left|\frac{2(\delta/a)\cos\gamma}{2-\zeta}\right| < 1$$

for all real values of γ, $0 \leqq \gamma \leqq \pi$, and for all complex values of δ/a and ζ in the domains defined by

$$0 \leqq |\zeta| \leqq 3/2\ ,$$

$$0 \leqq |\delta|/a < 1/4\ .$$

It follows immediately from equation (I.19) that $G[\zeta;(\delta/a)]$ is an infinitely differentiable function of γ and a regular function of δ/a and of ζ, when γ, $|(\delta/a)|$ and $|\zeta|$ are limited to the ranges

$$(I.22)\qquad \begin{aligned} &0 \leqq \gamma \leqq \pi,\\ &0 \leqq |\delta/a| \leqq \Delta''/a < \min\left\{1/4\ ,\ \Delta/a\right\} \equiv \Delta'/a,\\ &0 \leqq |\zeta-1| \leqq 1+\epsilon\ ,\quad \mathrm{Im}\,(\zeta-1) \geqq 0\ ,\end{aligned}$$

where

$$\epsilon = \Delta'/a - \Delta''/a \ ,$$

(see Figure 4).

If we operate exclusively within these ranges[*], the numerators in the integrals of (I.18) are both regular functions of δ/a and ζ and are infinitely differentiable functions of γ, $0 \leqq \gamma \leqq \pi$. The problem of ascertaining the behavior of I_f^+, therefore, reduces to determining the behavior of an integral of the form

$$\text{(I.24)} \qquad J^+ = \int_P \frac{F[\zeta;(\delta/a)]}{\sqrt{\zeta}\ \sqrt{2-\zeta}}\, d\zeta \ .$$

Here, $F[\zeta;(\delta/a)]$ is a regular function of the variables ζ and (δ/a) and is an infinitely differentiable function of γ, whenever these variables are confined to the above ranges.

To determine the behavior of J^+, we make use of Cauchy's theorem and find, since $F[\zeta;(\delta/a)]$ is regular within the semi-circle $|\zeta-1| \leqq 1+\epsilon$, $\operatorname{Im}\zeta \geqq 0$, that

$$\text{(I.25)} \qquad \int_P \frac{F[\zeta;(\delta/a)]}{\sqrt{\zeta}\ \sqrt{2-\zeta}}\, d\zeta = \int_{P_1+P_\epsilon'-P_2-P_3} \frac{F[\zeta;(\delta/a)]}{\sqrt{\zeta}\ \sqrt{2-\zeta}}\, d\zeta$$

where the parametric equations of the paths P_1, P_ϵ', P_2 and P_3, shown in Figure 4, are the following:

$$P_1 : \zeta = \mu\,(\delta/a)\, e^{i\gamma} \ , \quad \epsilon' \leqq \mu \leqq 1 \ ;$$

$$P_\epsilon' : \zeta = \epsilon'\, e^{i\sigma} \ , \quad 0 \leqq \sigma \leqq \pi \ ;$$

---

[*] It is assumed throughout that the branch points $\zeta = 0$ and $\zeta = 1$ have been excluded in the usual manner (see Figure 4).

$$P_2 : \zeta = \mu , \qquad \epsilon' \leqq \mu \leqq 1 ;$$

$$P_3 : \zeta = 1 + \mu(\delta/a)e^{i\gamma} , \quad 0 \leqq \mu \leqq 1 .$$

Evaluating the right member of (I.25) on each of these paths separately, and letting ϵ' approach zero, we find that

$$(I.26) \qquad J^+ = \int_P \frac{F[\zeta;(\delta/a)]}{\sqrt{\zeta}\ \sqrt{2-\zeta}}\, d\zeta = \sqrt{2}\ (\frac{\delta}{a})^{\frac{1}{2}}\ e^{i\gamma/2}\ F[0;0]$$

$$- \int_0^1 \frac{F[\zeta;0]}{\sqrt{\zeta}\,\sqrt{2-\zeta}}\, d\zeta + \mathcal{O}\,[(\delta/a)] \ ,$$

where we have exhibited explicitly only the first two terms of lowest order. The integral term in the right member must be interpreted as an improper integral. The first term in the right member comes from the integral over the path P_1. Combining this result and equation (I.18), we find that

$$(I.27) \qquad I_f^+ = - \frac{\sqrt{2}}{a}\ (\frac{\delta}{a})^{\frac{1}{2}}\ e^{i\gamma/2}\ f(1) + \frac{1}{a} \int_0^1 \frac{\exp(i\alpha\sqrt{\zeta}\ \sqrt{2-\zeta})f(1-\zeta)}{\sqrt{\zeta}\ \sqrt{2-\zeta}}\, d\zeta + \mathcal{O}\,[(\delta/a)]$$

which, on introducing $s = 1-\zeta$ as a new variable of integration, reduces to the result quoted in equation (I.5).

To determine the behavior of $\partial I_f^+/\partial\delta$, we operate on both sides of (I.18) with $\partial(\ )/\partial\delta$ and find that

$$(\mathrm{I}.28)\quad \frac{\partial I_f^+}{\partial \delta} = -\frac{e^{i\gamma}}{a^2}\left[\frac{\exp(i\alpha\sqrt{\zeta}\,\sqrt{2-\zeta})f(1-\zeta)}{\sqrt{\zeta}\,\sqrt{2-\zeta}}\Bigg|_{\zeta=1+\frac{\delta}{a}e^{i\gamma}}^{\zeta=\frac{\delta}{a}e^{i\gamma}}\right] + \left(\frac{\delta}{a}\right)\frac{e^{i\gamma}}{a}\left[\frac{G[\zeta;(\delta/a)]}{\sqrt{\zeta}\,\sqrt{2-\zeta}}\Bigg|_{\zeta=1+\frac{\delta}{a}e^{i\gamma}}^{\zeta=\frac{\delta}{a}e^{i\gamma}}\right]$$

$$+\frac{1}{a}\int_P \frac{G[\zeta;(\delta/a)]}{\sqrt{\zeta}\,\sqrt{2-\zeta}}\,d\zeta + \left(\frac{\delta}{a}\right)\int_P \frac{\partial G[\zeta;(\delta/a)]/\partial\delta}{\sqrt{\zeta}\,\sqrt{2-\zeta}}\,d\zeta\ .$$

The first line of this expression comes from the end points of P, whose position depends on δ/a.

Observe that the two integrals in the right member of equation (I.28) are of the same form as J^+. Since $J^+ = \mathcal{O}(1)$, we conclude that the last two terms of (I.28) are $\mathcal{O}(1)$ and $\mathcal{O}(\delta/a)$, respectively. If, now, in (I.28) we exhibit only the lowest order term explicitly, we find that

$$\frac{\partial I_f^+}{\partial\delta} = -\frac{f(1)}{\sqrt{2}\,a^2}\left(\frac{\delta}{a}\right)^{-\frac{1}{2}} e^{i\gamma/2} + \mathcal{O}(1)$$

which is the desired result [cf. (I.6)].

The behavior of $\delta^{-1}\partial I_f^+/\partial\gamma$ near the circular edge may be determined in a similar manner.

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